COMITATO NAZIONALE PER L'ENERGIA NUCLEARE
Laboratori Nazionali di Frascati

LNF-63/61
5. 9. 1963.

M. Ademollo and R. Gatto:

COMPLETE SPIN TESTS FOR FERMIONS. -

Nota interna: n. 211

Servizio Documentazione
dei Laboratori Nazionali di Frascati del CNEN
Casella Postale 70 - Frascati (Roma)

Laboratori Nazionali di Frascati del CNEN
Servizio Documentazione

LNF-63/61

Nota interna: n° 211
5 Settembre 1963

M. Ademollo[x] and R. Gatto: COMPLETE SPIN TESTS FOR FERMIONS.

SUMMARY.

The complete spin tests for fermions of arbitrary spin, produced from a spin zero boson on an unpolarized spin 1/2 fermion and decaying into a spin zero boson and a spin 1/2 fermion are derived. The tests constitute the set of necessary and sufficient conditions for the particular spin assignment, in the absence of more detailed dynamical information. Essential use is made of the R-invariance of the production process (following from parity conservation). More general tests, applicable to arbitrary production processes, are also discussed.

1. INTRODUCTION.

It is the main purpose of this note to derive the necessary and sufficient conditions for a spin assignment to a fermion produced from a spin-zero boson on an unpolarized spin 1/2 fermion and subsequently decaying into a spin 1/2 fermion and a spin zero boson. Our conclusions for this case are summarized in section 4.2 in a form allowing their direct practical use.

(x) - Istituto di Fisica dell'Università di Firenze.

We also discuss in this note spin tests applicable to more general production processes. Our conclusions for the general case are summarized in section 4.1.

The derivation of the necessary and sufficient conditions for the case of production from spin-zero boson on unpolarired spin 1/2 fermion is made possible by the application of the R-transformation to the production density matrix. Such a method was recently applied by Peshkin (1), and is also related to work by A. Bohr (2) and by Eberhard and Good (3). We obtain the necessary and sufficient conditions as a consequence of a theorem (shown in the text) that characterizes the most general density matrix for production of a fermion of spin s from two incoherent helicity states related by an R-transformation.

For the general production process we obtain various tests, some of them involving the longitudinal and transverse polarizations. These results are closely related to results by Lee and Yang (4), Durand, Landovitz and Leitner (5), by Spitzer and Stapp (6), by Gatto and Stapp (7), by Capps (8), by Ademollo and Gatto (9) and by Byers and Fenster (10).

2. THE DENSITY MATRIX FOR PRODUCTION FROM SPIN ZERO BOSON ON UNPOLARIZED SPIN 1/2 FERMION.

2.1 We consider the production process

$$a + f \longrightarrow F + \ldots \qquad (2.1)$$

where a is a spin-zero boson, f is a spin 1/2 fermion, and F is a fermion of spin s. The production process (2.1) is assumed to be parity conserving.

We assume that reaction (2.1) occurs on unpolarized f. Initially one thus has an equal-weight incoherent mixture of two helicity states.

As pointed out by Peshkin (1) the two initial helicity states are related to each other by an R-transformation. The transformation R is defined as a reflection through the production plane, or, equivalently, as the product of space inversion and rotation of 180 degrees around the normal to the production plane.

Each of the two initial helicity states gives rise to a final amplitude. Since parity is conserved in the production process also the two final amplitudes are related to each other by an R-transformation.

The density matrix of the final F will thus be of the form

$$\rho^{(F)} = \frac{1}{2}|\psi\rangle\langle\psi| + \frac{1}{2}|R\psi\rangle\langle R\psi| \qquad (2.2)$$

The final amplitudes ψ and $R\psi$ are not in general orthogonal. Following Peshkin (1) we take the normal $\vec{n}$ to the production plane as the direction of spin quantization and denote by μ the component of the spin of F along $\vec{n}$. We expand $|\psi\rangle$ in terms of the spin eigenfunctions $|\psi_\mu\rangle$

(2.3) $$|\psi\rangle = \sum_\mu \beta_\mu |\psi_\mu\rangle$$

with the normalization

(2.4) $$\sum_\mu |\beta_\mu|^2 = 1$$

Apart from an unimportant phase-factor we have

(2.5) $$|R\psi\rangle = \sum_\mu (-1)^{s-\mu} \beta_\mu |\psi_\mu\rangle$$

as can easily be seen from the interpretation of R as a space inversion followed by a rotation of π around $\vec{n}$. From (2.2), (2.3) and (2.5) we obtain

(2.6) $$\rho^{(F)} = \sum_{\mu\mu'} \frac{1}{2}\left[1+(-1)^{\mu-\mu'}\right]\beta_\mu\beta^*_{\mu'}|\psi_\mu\rangle\langle\psi_{\mu'}|$$

2.2 We see from Eq. (2.6) that $\rho^{(F)}$ differs from the density matrix that one would have obtained starting from a well--defined initial helicity state simply in that only the terms with $\mu-\mu'$ = even are kept in the summation in Eq. (2.6).

In other terms: the density matrix of F, produced according to (2.1) from unpolarized f, can be obtained from the density matrix for production from f in a well-defined helicity state (such a density matrix would thus describe a pure state) simply by suppressing all the matrix elements with $\mu-\mu'$ = odd.

The density matrix $\rho^{(F)}$, given by (2.6), corresponds to an incoherent mixture of two pure states. It does not therefore satisfy the limitation $\rho^2=\rho$ typical of a pure state. We have seen however that it can be obtained from the density matrix of a pure state by suppressing from it the matrix elements with $\mu-\mu'$ = odd. We shall now obtain the necessary and sufficient conditions that $\rho^{(F)}$ has to satisfy in order to be of such a form. These conditions will be used to formulate the complete set of conditions for spin s of F.

2.3 We first note that the conditions

(2.7) $$\rho^{(F)+} = \rho^{(F)} \quad \text{and} \quad \mathrm{Tr}\left[\rho^{(F)}\right] = 1$$

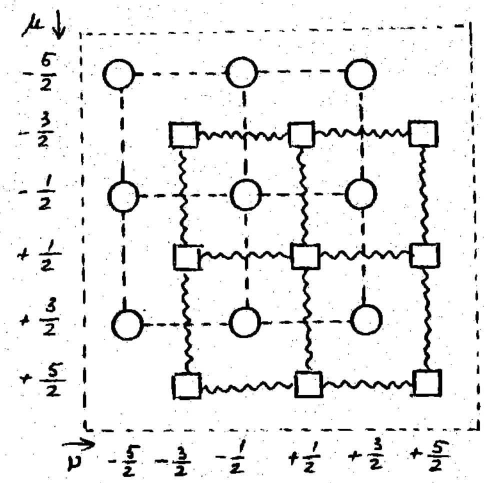
are identically satisfied by $\rho^{(F)}$, as can be seen from (2.6) and (2.4). We now split $\rho^{(F)}$ into the sum of two matrices ρ' and ρ'' defined as follows: ρ' has non-zero elements $\rho'_{\mu\nu} = \rho^{(F)}_{\mu\nu}$ only for $s-\mu$ and $s-\nu$ both odd; ρ'' has non-zero elements $\rho''_{\mu\nu} = \rho^{(F)}_{\mu\nu}$

only for $s-\mu$ and $s-\nu$ both even. We know that the matrix elements $\rho^{(F)}_{\mu\nu}$ with $s-\mu$ even (odd) and $s-\nu$ odd (even) are zero. Therefore

$$\rho^{(F)} = \rho' + \rho'' \tag{2.8}$$

where ρ' has the odd rows and colums of $\rho^{(F)}$, and ρ'' only the even rows and colums.

For instance, for spin s = 5/2, the matrices $\rho^{(F)}_{\mu\nu}$, $\rho'_{\mu\nu}$ and $\rho''_{\mu\nu}$ have dimensions 2s + 1 = 6. The indices μ, ν take the values -5/2, -3/2, -1/2, + 1/2, +3/2, and +5/2. The matrix $\rho^{(F)}$ has the form



Only the elements indicated by little circles or little squares can be different from zero. The matrix elements indicated by little circles form the matrix ρ' ; those indicated by little squares form the matrix ρ''.

Turning to the general discussion, we note that the matrices ρ' and ρ'' are related by the condition

$$Tr[\rho'] + Tr[\rho''] = Tr[\rho^{(F)}] = 1 \tag{2.9}$$

2.4 To ρ' and ρ'' applies the following theorem: Let ρ be an hermitian matrix with trace a

$$\rho^{+} = \rho \quad ; \qquad Tr[\rho] = a \tag{2.10}$$

The necessary and sufficient conditions in order that ρ be of the form

$$\rho = \beta\beta^{+} \tag{2.11}$$

where β is a one-colum matrix, are:

i) $a \geq 0$

ii) ρ has characteristic one.

It is easy to see that the conditions are necessary. In fact from (2.11) it follows that:

(2.12) $$a = Tr[\rho] = \sum_\mu |\beta_\mu|^2 \geq 0$$

(2.13) $$\rho_{\mu\nu}\rho_{\nu\tau} - \rho_{\mu\tau}\rho_{\sigma\nu} = 0$$

i.e. all minors of order two are zero.

To prove the sufficiency let us suppose that (2.12) and (2.13) are satisfied. From (2.13) putting $\sigma=\nu$ and summing over ν we obtain

(2.14) $$\rho^2 = a\rho \ ,$$

Let now u be an unitary matrix that brings ρ in the diagonal form $\tilde{\rho}$

(2.15) $$u^{-1}\rho u = \tilde{\rho}$$

From (2.14) we see that the eigenvalues of ρ are 0 or a. However only one eigenvalue a may exist because of the second of (2.10). The diagonal matrix $\tilde{\rho}$ has thus only one element different from zero. We can suppose this element (whose value is a) to be the one on the first row and on the first column. Using (2.15), (2.14) and (2.12) we have, for a≠0 (otherwise ρ vanishes identically)

(2.16) $$\rho = u\tilde{\rho}u^{-1} = \frac{1}{a}u\tilde{\rho}\tilde{\rho}u^{+} = (a^{-\frac{1}{2}}u\tilde{\rho})(a^{-\frac{1}{2}}u\tilde{\rho})^{+}$$

Since $a^{-\frac{1}{2}}u\tilde{\rho}$ has only the first column different from zero, it can be identified with the one-column matrix β of (2.11).

We have thus proved the theorem.

2.5 From the proof of the theorem, as given above, it follows that the condition (2.14) is equivalent to (2.13).

Also we note that taking Eq. (2.13) with $\mu=\nu$ and $\tau=\sigma$ we find

(2.17) $$\rho_{\mu\mu}\rho_{\sigma\sigma} = \rho_{\mu\sigma}\rho_{\sigma\mu} = |\rho_{\mu\sigma}|^2$$

the last step following from the hermiticity condition on ρ. Eq. (2.17) tells us that all non-zero diagonal elements of ρ have the same sign. The condition (2.12), that the trace be positive, can thus be substituted by the condition that one non-zero diagonal matrix elements of ρ be positive.

If a matrix ρ is of the form (2.11) it must clearly be true that

(2.18) $$\rho_{\mu_1\nu_1}\rho_{\mu_2\nu_2}\cdots\rho_{\mu_n\nu_n} = \rho_{\mu_1\nu_{i_1}}\rho_{\mu_2\nu_{i_2}}\cdots\rho_{\mu_n\nu_{i_n}}$$

ly present) will be considered elsewhere.

3. THE DECAY DENSITY MATRIX. CONSTRUCTION OF THE PRODUCTION DENSITY MATRIX.

3.1 We assume that the fermion F produced in a general production process

(3.1) $$A + B \longrightarrow F + \dots$$

decays according to

(3.2) $$F \longrightarrow f' + c$$

where f' is a spin $\frac{1}{2}$ fermion and c is a spin zero boson. The decay process (3.2) is not assumed to be parity-conserving. The analysis can thus be applied either to a weakly decaying F (for instance Ξ production and subsequent decay) or to a strongly decaying F (production and decay of baryon resonant states).

The production process (3.1) is a general production process from two particles. It can in particular occur from a spin zero-boson colliding with an unpolarized spin $\frac{1}{2}$ fermion, as assumed in the preceding section. Some of the results of this section 3 and of the next section 4 are more general and they apply to any production process. We thus consider a general production process of the kind (3.1) for F.

The most general density metrix of F can be written as

$$\rho^{(F)} = \sum_{\mu\mu'} \rho_{\mu\mu'} |\psi_\mu\rangle\langle\psi_{\mu'}|$$

By specialising $\rho_{\mu\mu'}$ as

$$\rho_{\mu\mu'} = \frac{1}{2}\left[1 + (-1)^{\mu-\mu'}\right]\beta_\mu \beta^*_{\mu'}$$

one has the matrix (2.6), typical of the special production process (2.1) (boson on unpolarized spin $\frac{1}{2}$ fermion).

We call $\vec{u}$ and $\vec{u}'$, respectively the unit vectors along the momenta of B and F in the center of mass frame of reaction (3.1); $\vec{n} = (\vec{u} \times \vec{u}')/|\vec{u} \times \vec{u}'|$ is the normal to the production plane, and we call $\vec{v}$ the unit vector along the momentum of f' in the rest system of F.

After F decays the eigenfunction $|\psi_\mu\rangle$, in the general expression for $\rho^{(F)}$, becomes, in its angular and spin dependence:

(3.3) $$|\psi_\mu\rangle \rightarrow \sum_{lm} (-1)^l T_l(lm, \tfrac{1}{2}\nu|s\nu) Y_l^m(\vec{v})|\chi_\nu\rangle$$

where T_ℓ are the decay matrix elements, $(\ell m, \frac{1}{2}\nu/s\mu)$ are the usual Clebsch-Gordan-Wigner coefficents and $|\chi_\nu\rangle$ are the spin eigenstates of f'. We have explicitly introduced a factor $(-1)^l$ because our angular distribution refers to the final fermion f', instead of the final boson c.

From (3.3) we obtain, for the final state density matrix

$$\rho^{(f)} = \sum_{\mu\mu'} \rho_{\mu\mu'} \sum_{\ell\ell'} (-1)^{\ell-\ell'} T_\ell T^*_{\ell'} \sum_{mm'\nu\nu'} (\ell m, \tfrac{1}{2}\nu/s\mu)\cdot Y_\ell^m(\vec{v}) \cdot (\ell' m', \tfrac{1}{2}\nu'/s\mu') Y_{\ell'}^{m'*}(\vec{v}) |\chi_\nu\rangle\langle\chi_{\nu'}| . \tag{3.4}$$

We can now calculate the angular distribution of f', I(v), and its polarization $\vec{P}$, using the formulas

$$I(\vec{v}) = Tr\, \rho^{(f)} \tag{3.5}$$

$$I(\vec{v})\vec{P} = Tr\, \rho^{(f)}\vec{\sigma} , \tag{3.6}$$

The explicit calculations are briefly outlined in the Appendix. The results are the following:

$$I(v) = \sum_{L,M} a(L,M)\, Y_L^M(\vec{v}) \tag{3.7}$$

with

$$a(L,M) = (-1)^{s-\frac{1}{2}} \sqrt{\frac{2s+1}{4\pi}}\, (s\tfrac{1}{2}, s-\tfrac{1}{2}/L0) \sum_\mu \rho_{\mu,\mu-M} (s\mu-M, LM/s\mu) \tag{3.7'}$$

for even L and

$$a(L,M) = (-1)^{s-\frac{1}{2}} \alpha \sqrt{\frac{2s+1}{4\pi}}\, (s\tfrac{1}{2}, s-\tfrac{1}{2}/L0) \sum_\mu \rho_{\mu,\mu-M} (s\mu-M, LM/s\mu), \tag{3.7''}$$

for odd L. For the longitudinal polarization we have:

$$I(\vec{v})\vec{P}\vec{v} = \sum_{L,M} b_1(L,M)\, Y_L^M(\vec{v}) \tag{3.8}$$

where

$$b_1(L,M) = (-1)^{s-\frac{1}{2}} \alpha \sqrt{\frac{2s+1}{4\pi}}\, (s\tfrac{1}{2}, s-\tfrac{1}{2}/L0) \sum_\mu \rho_{\mu,\mu-M} (s\mu-M, LM/s\mu) \tag{3.8'}$$

for even L and

$$b_1(L,M) = (-1)^{s-\frac{1}{2}} \sqrt{\frac{2s+1}{4\pi}}\, (s\tfrac{1}{2}, s-\tfrac{1}{2}/Lo) \sum_\mu \rho_{\mu,\mu-M} (s\mu-M, LM/s\mu) \tag{3.8''}$$

for odd L. The polarization along $\vec{n} \times \vec{v}$ is given by

$$I(\vec{v})\, \vec{P}.\vec{n}\times\vec{v} = \sum_{L,M} b_2(L,M)\, Y_L^M(\vec{v}) \tag{3.9}$$

where

$$b_2(L,M) = i\epsilon \sqrt{\frac{2s+1}{4\pi}}\, \frac{2s+1}{\sqrt{L(L+1)}}\, (s\tfrac{1}{2}, s-\tfrac{1}{2}/L0)(LM, 10/LM) \cdot \sum_\mu \rho_{\mu,\mu-M} (s\mu-M, LM/s\mu) \tag{3.9'}$$

for odd L and, for even L:

(3.9")
$$b_2(L,M) = (-1)^{s-\frac{1}{2}} \beta \sqrt{\frac{2s+1}{4\pi}} \frac{2s+1}{\sqrt{2L+1}} \sum_{\text{odd } f} C_f \, (s\tfrac{1}{2}, s-\tfrac{1}{2}/f0) \cdot$$
$$\cdot (fM, 10/LM) \sum_{\mu} \rho_{\mu,\mu-M} (s\mu-M, fM/s\mu)$$

where

(3.9"')
$$C_f = \begin{cases} (f+1)^{-\frac{1}{2}} & \text{for } f = L-1 \\ f^{-\frac{1}{2}} & \text{for } f = L+1 \end{cases}$$

Finally, for the polarization along $\vec{v} \times (\vec{n} \times \vec{v})$ we have

(3.10)
$$I(\vec{v})\, \vec{P} \cdot \vec{v} \times (\vec{n} \times \vec{v}) = \sum_{L,M} b_3(L,M)\, Y_L^M(\vec{v})$$

where

(3.10')
$$b_3(L,M) = (-1)^{s-\frac{1}{2}} i\beta \sqrt{\frac{2s+1}{4\pi}} (s\tfrac{1}{2}, s-\tfrac{1}{2}/L0)(LM, 10/LM) \cdot$$
$$\cdot \sum_{\mu} \rho_{\mu,\mu-M} (s\mu-M, LM/s\mu)$$

for odd L and, for even L:

(3.10")
$$b_3(L,M) = -\epsilon \sqrt{\frac{2s+1}{4\pi}} \frac{2s+1}{\sqrt{2L+1}} \sum_{\text{odd } f} C_f \, (s\tfrac{1}{2}, s-\tfrac{1}{2}/f0) \cdot$$
$$\cdot (fM, 10/LM) \sum_{\mu} \rho_{\mu,\mu-M} (s\mu-M, fM/s\mu)$$

where C_f is given by (3.9"').

We have normalized the matrix elements T_ℓ such that

(3.11)
$$|T_{s-\frac{1}{2}}|^2 + |T_{s+\frac{1}{2}}|^2 = 1$$

so that the angular distribution $I(\vec{v})$ results normalized to unity in the whole solid angle. We have also used the following notations:

(3.12)
$$\alpha = 2 \operatorname{Re} T_{s-\frac{1}{2}} T^*_{s+\frac{1}{2}}$$

(3.12')
$$\beta = 2 \operatorname{Im} T_{s-\frac{1}{2}} T^*_{s+\frac{1}{2}}$$

(3.12")
$$\epsilon = (-1)^{s-\frac{1}{2}} [|T_{s-\frac{1}{2}}|^2 - |T_{s+\frac{1}{2}}|^2]$$

we have

(3.12"')
$$\alpha^2 + \beta^2 + \epsilon^2 = 1 .$$

3.2 If parity is conserved in reaction (3.2) α and β are zero and ϵ is +1 for final l-wave even and -1 fro final l-wave odd.

We remark that if the production process is of the special form (2.1) on unpolarized f, in all the above relations we have

$$M = \mu - \mu' = \text{even}$$

Contributions from odd M vanish because of the special form of the density matrix $\rho^{(F)}$ as given by Eq. (2.6).

We observe that in the Adair's limit of 0° or 180° production angle (11), we must have symmetry about the direction of the incident beam, that is M=0 and $\rho_{\mu\mu} = \rho_{-\mu-\mu}$. In this case the angular distribution contains only the even L terms:

(3.13) $$I(\vec{v}) = \sum_{\text{even } L} a(L,0)\, Y_L^0(\vec{v})$$

and the polarization $\vec{P}$ is completely longitudinal and constant:

(3.14) $$\vec{P} = \alpha \vec{v}$$

3.3 The coefficients a(L,M) and b_i(L,M) can be obtained from experiment by certain weighted averages of the angular distribution and polarization of the final fermion. Precisely

(3.15) $$a(L,-M) = \langle Y_M^L \rangle$$

(3.16) $$b_1(L,-M) = \langle \vec{P}\cdot\vec{v}\, Y_L^M \rangle$$

(3.17) $$b_2(L,-M) = \langle \vec{P}\cdot(\vec{n}\times\vec{v})\, Y_L^M \rangle$$

(3.18) $$b_3(L,-M) = \langle \vec{P}\cdot\vec{v}\times(\vec{n}\times\vec{v})\, Y_L^M \rangle$$

where

$$\langle A \rangle = \int I(\vec{v})\, A(\vec{v})\, d\Omega_{\vec{v}}$$

Owing to the reality of $I(\vec{v})$ we have:

$$a(L,-M) = (-1)^M a(L,M)$$

and analogous relations for b_i(L,M)

From (3.7')-(3.10") we obtain the following general relations among the coefficients a(L,M) and b_i(L,M):

(3.19) $$b_1(L,M) = \alpha\, a(L,M) \qquad \text{for even } L$$

(3.19') $$b_1(L,M) = \frac{1}{\alpha}\, a(L,M) \qquad \text{for odd } L$$

(3.20) $$b_2(L,M) = (-1)^{s-\frac{1}{2}} \frac{i\epsilon}{\alpha} \frac{2s+1}{L(L+1)} M a(L,M) \qquad \text{for odd } L$$

(3.20') $$b_2(L,M) = \frac{\beta}{\alpha}(2s+1)\Big[\frac{1}{L}\sqrt{\frac{(L-M)(L+M)}{(2L-1)(2L+1)}}\, a(L-1,M) - \frac{1}{L+1}\sqrt{\frac{(L+1-M)(L+1+M)}{(2L+1)(2L+3)}}\, a(L+1,M)\Big] \qquad \text{for even } L$$

(3.21) $$b_3(L,M) = \frac{i\beta}{\alpha} \frac{2s+1}{L(L+1)} M a(L,M) \qquad \text{for odd } L$$

$$(3.21') \qquad b_3(L,M) = (-1)^{s+\frac{1}{2}} \frac{\epsilon}{\alpha}(2s+1)\Big[\frac{1}{L}\sqrt{\frac{(L-M)(L+M)}{(2L-1)(2L+1)}}\, a(L-1,M) -$$

$$- \frac{1}{L+1}\sqrt{\frac{(L+1-M)(L+1+M)}{(2L+1)(2L+3)}}\, a(L+1,M)\Big] \qquad \text{for even } L$$

and also

$$(3.22) \qquad b_3(L,M) = (-1)^{s-\frac{1}{2}} \frac{\beta}{\epsilon}\, b_2(L,M) \qquad \text{for odd L}$$

$$(3.22') \qquad b_3(L,M) = (-1)^{s+\frac{1}{2}} \frac{\epsilon}{\beta}\, b_2(L,M) \qquad \text{for even L}$$

which are equivalent to (3.21) and (3.21') respectively

For L = 0 in the r.h.s. of (3.20') and (3.21') only the second term in the square brackets exists.

All the above relations are valid for any production process of F.

These relations can furnish a general test for determining the spin and decay parameters of F, provided only that the transverse polarization is appreciable. In fact, fitting these relations for various values of L and M, we obtain

$$\alpha; \quad \epsilon' = (-1)^{s-\frac{1}{2}}(2s+1)\epsilon; \quad \beta' = (2s+1)\beta$$

as parameters. Using (3.12"') we obtain

$$(3.23) \qquad \beta'^2 + \epsilon'^2 = (2s+1)^2(1-\alpha^2)$$

which gives unambigously the spin s (which in turn permits the determination of β and ϵ) if $\beta, \epsilon \neq 0$ and $\alpha \neq \pm 1$.

Other tests given in the following are valid for this particular situation.

As a particular case, using (3.20') and (3.21') with L = M = 0 and also (3.19') and the definitions (3.15) -(3.18) we have, for any value of α :

$$\beta' = -\frac{\langle \vec{P}\cdot(\vec{n}\times\vec{v})\rangle}{\langle(\vec{P}\cdot\vec{v})(\vec{n}\cdot\vec{v})\rangle}$$

$$\epsilon' = \frac{\langle \vec{P}\cdot\vec{v}\times(\vec{n}\times\vec{v})\rangle}{\langle(\vec{P}\cdot\vec{v})(\vec{n}\cdot\vec{v})\rangle}$$

and (3.23) can be written

$$(3.24) \qquad (2s+1)^2 = \frac{\langle \vec{P}\cdot(\vec{n}\times\vec{v})\rangle^2 + \langle \vec{P}\cdot\vec{v}\times(\vec{n}\times\vec{v})\rangle^2}{\langle(\vec{P}\cdot\vec{v})(\vec{n}\cdot\vec{v})\rangle^2 - \langle \vec{n}\cdot\vec{v}\rangle^2}$$

This relation provides a very direct general test for the spin s.

3.4 Let us consider in detail the case in which the particle F is produced in reaction (2.1) that is in a strong interaction between a particle of spin 0 and an unpolarized particle of spin $\frac{1}{2}$.

Eqs. (3.15), (3.7') and (3.7") can be written as

(3.25) $$\langle Y_L^M \rangle = (-1)^{s-\frac{1}{2}} \sqrt{\frac{2s+1}{4\pi}}\, g_L (s\tfrac{1}{2}, s-\tfrac{1}{2}/L0) \sum_\mu \rho_{\mu,\mu+M} (s\mu, LM/s\mu+M)$$

where M is even and

$$g_L = \frac{1}{2}\left[(1+\alpha)+(-1)^L(1-\alpha)\right] = \begin{cases} 1 & \text{for even } L \\ \alpha & \text{for odd } L \end{cases}$$

Eq. (3.25) can be inverted giving:

(3.26) $$\rho_{\mu,\mu+M} = (-1)^{\mu-\frac{1}{2}} \frac{\sqrt{4\pi}}{2s+1} \sum_L \frac{1}{g_L} \sqrt{2L+1}\, \frac{(s\mu, s-\mu-M/L-M)}{(s\tfrac{1}{2}, s-\tfrac{1}{2}/L0)} \langle Y_L^M \rangle$$

We now discuss a possible way of using this relation to test the spin of F.

Determining from the experiment the averages $\langle Y_L^M \rangle$ for a fixed value of s we can construct the matrix $\rho^{(F)}$ whose elements $\rho^{(F)}_{\mu\nu}$ are given by the r.h.s. of (3.26) and where $\nu = \mu + M$. If $\alpha = 0$, for the odd-L terms we can substitute the undetermined form $1/\alpha \langle Y_L^M \rangle$ by $\langle \vec{P}.\vec{v}\, Y_L^M \rangle$, using (319').

Now $\rho^{(F)}$ must be the F density matrix (2.6) and its matrix elements must be of the form

(3.27) $$\rho^{(F)}_{\mu\nu} = \beta_\mu \beta^*_\nu \;; \qquad \nu - \mu = M = \text{even}$$

We easily see that the conditions

$$\rho^{(F)+} = \rho^{(F)} \;; \qquad Tr\left[\rho^{(F)}\right] = 1$$

are identically satisfied by (3.26).

We have shown, in section (2.4), that the necessary and sufficient conditions in order that $\rho^{(F)}$ be the density matrix of F, produced according to reaction (2.1) on unpolarized f, are that, after decomposing $\rho^{(F)}$ according to Eq. (2.8), both ρ' and ρ'' have characteristic one, and their traces be non-negative.

4. CONCLUSIONS.

We now summarize the possible tests for the determination of the spin of the particle F.

4.1 General tests, valid for any production process, are those considered by Lee and Yang (4) and by Durand, Landovitz and Leitner (5), which follow from the conditions

(4.1) $$\rho^{(F)}_{\mu\mu} \geq 0 \;; \qquad \sum_\mu \rho^{(F)}_{\mu\mu} = 1$$

$\rho^{(F)}_{\mu\mu}$ is in fact the probability P_μ to find F with spin component μ along any direction $\vec{n}$.

The tests we shall consider here are inequalities, following from (4.1), which constitute necessary conditions for the spin assignment.

We have two kinds of such inequalities. First we have limitations on the experimental averages (3.15)-(3.18) for M = 0. In fact all the expressions of a (L,M) and $b_i(L,0)$ contain the factor

$$\sum_{\mu} \rho_{\mu\mu}(s\mu, L0/s\mu)$$

which is the expectation value of the Clebsch-Gordan coefficient $(s\mu, L0/s\mu)$ in the initial state. Thus we have

$$(4.2) \qquad \left|\sum_{\mu} \rho_{\mu\mu}(s\mu, L0/s\mu)\right| \leq \operatorname{Max}\left|(s\mu, L0/s\mu)\right|$$

where the notation Max on the r.h.s. of (4.2) means the maximum value of the argument, with respect to μ.

For the angular distribution we have:

$$(4.3) \qquad \left|\langle P_L(\vec{n}\cdot\vec{v})\rangle\right| \leq \left|(s\tfrac{1}{2}, L0/s\tfrac{1}{2})\right| \operatorname{Max}\left|(s\mu, L0/s\mu)\right|$$

where we have used the Legendre polynomials P_L instead of the sherical harmonics; for odd L we have also made use of the condition $|\alpha| \leq 1$. For L = 1, being $\operatorname{Max}|(s\mu, 10/s\mu)| = (s/s+1)^{\frac{1}{2}}$ we obtain the well known limitations of Lee and Yang:

$$(4.4) \qquad \left|\langle \vec{n}\cdot\vec{v}\rangle\right| \leq \frac{1}{2s+2}$$

For the longitudinal polarization we have limitations analogous to (4.3) and (4.4):

$$(4.5) \qquad \left|\langle \vec{P}\cdot\vec{v}\; P_L(\vec{n}\cdot\vec{v})\rangle\right| \leq \left|(s\tfrac{1}{2}, L0/s\tfrac{1}{2})\right| \operatorname{Max}\left|(s\mu, L0/s\mu)\right|$$

which for L = 1 becomes

$$(4.6) \qquad \left|\langle(\vec{P}\cdot\vec{v})(\vec{n}\cdot\vec{v})\rangle\right| \leq \frac{1}{2s+2}$$

Analogous limitations can also be found for transverse polarization but they result less useful than the preceding. For example, from (3.9") and for L = 0, we obtain

$$(4.7) \qquad \left|\langle \vec{P}\cdot\vec{n}\times\vec{v}\rangle\right| \leq \frac{2s+1}{2s+2}$$

More general tests we can derive from (4.1) are limitations on some test functions that we can construct from the experimental averages.

We have, independent of the value of α:

$$(4.8) \qquad T_\mu = (-1)^{\mu-\frac{1}{2}} \frac{1}{2s+1} \sum_L (2L+1) \frac{(s\mu, s-\mu/L0)}{(s\tfrac{1}{2}, s-\tfrac{1}{2}/L0)} \langle P_L\rangle \geq 0$$

In fact the sum over the even-L terms gives $\frac{1}{2}(P_\mu + P_{-\mu})$ whereas the sum over the odd-L terms gives $\frac{\alpha}{2}(P_\mu - P_{-\mu})$. Since

$|\alpha| \leq 1$ the entire sum is positive.

We notice that limitations entirely analogous to (4.8) gold for the test functions obtained by the substitution in (4.8) or $\langle P_L \rangle$ with $\langle \vec{P}\cdot\vec{v}\, P_L \rangle$.

Analogous but a little more complicated test functions can be defined for the transverse polarization (See e.g. ref. 5).

We enphasize that limitations (4.3)-(4.7), or alternatively (4.8) and analogous, must be verified for any production process, but they are in general not sufficient for the spin determination. We also remark that such limitations do not involve any knowledge of the decay parameters α, β and ϵ.

More precise tests, also valid in the general case, are those discussed in subsection 3.3. They consist in verifying the relations (3.19)-(3.21) among the averages a (L,M) and b_i(L,M) defined as in (3.15)-(3.18). This would give the possibility of a direct determination of the spin and decay parameters of F (see e.g. Eq. (E.24)).

4.2 We now discuss the particular case of production of F from a spin zero-boson on an unpolarized spin $\frac{1}{2}$ fermion.

Complete tests for the spin of F can be performed in the following way:
i) From the experimental angular distribution $I(\vec{v})$ of f' in the decay of F according to reaction (3.2) one computes the averages

$$\langle Y_L^M \rangle = \int Y_L^M(\vec{v})\, I(\vec{v})\, d\Omega_{\vec{v}}$$

where $d\Omega_{\vec{v}}$ is the element of solid angle around $\vec{v}$, the unit vector along the momentum of f' in the rest system of F.
ii) One then constructs the test matrix

$$\rho_{\mu\nu} = \sum_L C_{\mu\nu}^{(s)}(L)\, \frac{1}{g_L} \langle Y_L^M \rangle \tag{4.9}$$

where $\nu = \mu + M$, M even, and

$$C_{\mu\nu}^{(s)}(L) = (-1)^{\mu-\frac{1}{2}}\, \frac{\sqrt{4\pi}}{2s+1} \sqrt{2L+1}\, \frac{(s\mu, s-\nu / L-M)}{(s\tfrac{1}{2}, s-\tfrac{1}{2} / L0)}, \tag{4.10}$$

The coefficients $C_{\mu\nu}^{(s)}$ (L) have been numerically evaluated for s = 1/2, 3/2 and 5/2. Their values are reported in tables Ia, Ib and Ic. In (4.9) g_L is given by

$$g_L = \begin{cases} 1 & \text{for even } L \\ \alpha & \text{for odd } L \end{cases}$$

The decay parameter α can be determined for instance from (3.19) as the average longitudinal polarization

(4.11) $$\alpha = \langle \vec{P} \cdot \vec{V} \rangle$$

iii) The test matrix $\rho_{\mu\nu}$, in order to be a possible density matrix for F produced from a spin-zero boson on an unpolarized spin $\frac{1}{2}$ fermion must be such that the two matrices $\rho'_{\mu\nu}$ (obtained from $\rho_{\mu\nu}$ by putting equal to zero the matrix elements for which s-μ and s-ν are both even) and $\rho''_{\mu\nu}$ (obtained from $\rho_{\mu\nu}$ by putting equal to zero the elements with s-μ and s-ν both odd) both have characteristic one, and non-negative traces. The condition on the characteristic can be verified by computing the $[(2s-1)/2]^2$ independent minors of order 2 for each of the two matrices ρ' and ρ'' and verifying that they be zero within experimental error. A convenient choice of the independent minors leads to the set of relations (in total $\frac{1}{2}(2s-1)^2$ relations)

(4.12) $$\rho'_{\alpha\alpha}\, \rho'_{\sigma\tau} = \rho'_{\alpha\tau}\, \rho'_{\sigma\alpha}$$

(4.13) $$\rho''_{\beta\beta}\, \rho''_{\sigma\tau} = \rho''_{\beta\tau}\, \rho''_{\sigma\beta}$$

where σ and τ take on all possible values different from α in (4.12) and all possible values different from β in (4.13). To the relations (4.12) and (4.13) one must add the conditions

(4.14) $$\Sigma\, \rho'_{\mu\mu} \geq 0 \quad \text{and} \quad \Sigma\, \rho''_{\mu\mu} \geq 0$$

(See subsection 4.1). Eqs. (4.12), (4.13) and the inequalities (4.14) provide a necessary and sufficient set of conditions for spin s of F.

Other procedures for checking the condition that ρ' and ρ'' have characteristic 1 can be derived by choosing differently the $[(2s-1)/2]^2$ independent minors of order 2 in both ρ' and ρ''. The most convenient procedure will presumably be suggested in each case by the numerical aspect of the matrix $\rho_{\mu\nu}$ determined empirically according to (4.9).

Of course the general tests discussed in section 4.1 for a general production process also apply to the case discussed in this section.

APPENDIX.

We shall briefly derive the general expressions of the coefficients a(L,M) and b_i(L,M) of Eqs. (3.7)-(3.10) using the standard Racah's algebra.

a) Angular distribution.

From (3.5) and (3.4) and using the formula

(A.1) $$Y_{\ell}^{m}(\vec{v})\, Y_{\ell'}^{m'*}(\vec{v}) = (-1)^{m'} \frac{\hat{\ell}\hat{\ell'}}{\sqrt{4\pi}} \sum_{L,M} \frac{1}{\hat{L}} (\ell 0, \ell' 0 | L 0)(\ell m, \ell' m' | L M)\, Y_L^{M}(\vec{v})$$

where we have used the abreviation

$$\hat{\ell} \equiv \sqrt{2\ell+1}$$

we obtain

(A.2) $$I(\vec{v}) = \sum_{LM} a(L,M)\, Y_L^M(\vec{v})$$

where

(A.2') $$a(L,M) = \frac{(-1)^L}{\sqrt{4\pi}\,\hat{L}} \sum_{\mu\mu'} \varrho_{\mu\mu'} \sum_{\ell\ell'} T_\ell T_{\ell'}^* \hat{\ell}\hat{\ell}'(\ell 0, \ell' 0/L0) \cdot \sum_{mm'\nu} (-1)^{m'} (\ell m, \tfrac{1}{2}\nu/s\mu)(\ell' m', \tfrac{1}{2}\nu/s\mu')(\ell m, \ell' -m'/LM)$$

Using the formula

(A.3) $$\sum_{\alpha\beta\varepsilon} (a\alpha, b\beta/e\varepsilon)(e\varepsilon, d\delta/c\gamma)(b\beta, d\delta/f\varphi) = \hat{e}\hat{f}\, W(abcd;ef)(a\alpha, f\varphi/c\gamma)$$

where W is the standard Racah coefficient, and the symmetry properties of the Clebch-Gordan coefficients, the sum over the magnetic quantum numbers in (A.2') becames

$$(-1)^{L+s-\frac{1}{2}}\, \hat{s}\hat{L}\, W(s\ell s\ell'; \tfrac{1}{2}L)(s\mu', LM/s\mu)$$

We now use the remarkable formula

(A.4) $$\hat{b}\hat{d}(b0, d0/f0)\, W(abcd; \tfrac{1}{2}f) = (a\tfrac{1}{2}, c-\tfrac{1}{2}/f0)$$

so that any dependence on l and l' disappears in the coefficient of $T_l\, T_{l'}^*$ in (A.2') and we obtain:

(A.5) $$a(L,M) = (-1)^{s-\frac{1}{2}} \frac{\hat{s}}{\sqrt{4\pi}} (s\tfrac{1}{2}, s-\tfrac{1}{2}/L0) \sum_{\ell\ell'} T_\ell T_{\ell'}^* \sum_{\mu\mu'} \varrho_{\mu\mu'} (s\mu', LM/s\mu)$$

Observing, now, that l and l' take the values s $\pm \frac{1}{2}$ and that l+l'+L must be even because of the C.G. coefficient $(\ell 0, \ell' 0/L0)$ in (A.2'), and remembering the definitions (3.11) and (3.12), we obtain from (A.5) the expressions (3.7') and (3.7") respectively for even and odd L.

b) Polarization.

Let us define three orthogonal unit vectors as

(A.6) $$\vec{v}_1 = \vec{v}\,; \qquad \vec{v}_2 = \frac{\vec{n} \times \vec{v}}{|\vec{n} \times \vec{v}|}\,; \qquad \vec{v}_3 = \vec{v}_1 \times \vec{v}_2$$

The polarization of the final fermion f' along the direction v_i is, by (3.6) and (3.4):

(A.7) $$I(\vec{v})\, \vec{P}\cdot\vec{v}_i = \sum_{\mu\mu'} \varrho_{\mu\mu'} \sum_{\ell\ell'} (-1)^{\ell-\ell'} T_\ell T_{\ell'}^* \sum_{mm'\nu\nu'} (\ell m, \tfrac{1}{2}\nu/s\mu) \cdot (\ell' m', \tfrac{1}{2}\nu'/s\mu')\, Y_\ell^m(\vec{v})\, Y_{\ell'}^{m'*}(\vec{v}) \langle \nu'/\vec{\sigma}\cdot\vec{v}_i/\nu \rangle .$$

It is easy to see that

(A.8) $$\langle \nu'/\vec{\sigma}\cdot\vec{v}_i/\nu \rangle = \sqrt{4\pi} \sum_x (\tfrac{1}{2}\nu', 1x/\tfrac{1}{2}\nu)\, Y_1^x(\vec{v}_i)$$

Furthermore we can express the spherical harmonics of argument $\vec{v}_2$ and $\vec{v}_3$ by those of argument $\vec{v}$:

(A.9) $$|\vec{n}\times\vec{v}|\, Y_1^{\chi}(\vec{v}_2) = i\sqrt{2}\,(1\chi, 10|1\chi)\, Y_1^{\chi}(\vec{v})$$

(A.9') $$|\vec{n}\times\vec{v}|\, Y_1^{\chi}(\vec{v}_3) = -\sqrt{\tfrac{4\pi}{3}}\, Y_1^{0}(\vec{v})\, Y_1^{\chi}(\vec{v}) + \sqrt{\tfrac{3}{4\pi}}\,\delta_{\chi 0}$$

Substituting (A.8), (A.9) and (A.9') in (A.7) and using suc cessivelly (A.1) we get

(A.10) $$I(\vec{v})\,\vec{P}\cdot\vec{v} = \sum_{LM} b_1(L,M)\, Y_L^M(\vec{v})$$

where

(A.10') $$b_1(L,M) = \sqrt{\tfrac{3}{4\pi}}\;\tfrac{1}{2}\sum_{\mu\mu'} \varrho_{\mu\mu'} \sum_{\ell\ell'} (-1)^{\ell-\ell'} T_\ell T_{\ell'}^{*}\, \hat{\ell}\hat{\ell}' \sum_{\ell''} (\ell 0, \ell' 0|\ell'' 0)\cdot$$
$$\cdot(\ell'' 0, 10|L0) \sum_{\substack{m m' m''\\ \nu\nu'\chi}} (-1)^{m'} (\ell m, \tfrac{1}{2}\nu|s\mu)(\ell' m', \tfrac{1}{2}\nu'|s\mu')\cdot$$
$$\cdot(\tfrac{1}{2}\nu', 1\chi|\tfrac{1}{2}\nu)(\ell m, \ell' {-m'}|\ell'' m'')(\ell'' m'', 1\chi|LM)$$

(A.11) $$I(\vec{v})\,\vec{P}\cdot(\vec{n}\times\vec{v}) = \sum_{LM} b_2(L,M)\, Y_L^m(\vec{v})$$

where

(A.11') $$b_2(L,M) = i\sqrt{\tfrac{3}{2\pi}}\;\tfrac{1}{2}\sum_{\mu\mu'} \varrho_{\mu\mu'} \sum_{\ell\ell'} (-1)^{\ell-\ell'} T_\ell T_{\ell'}^{*}\, \hat{\ell}\hat{\ell}' \sum_{\ell''} (\ell 0, \ell' 0|\ell'' 0)\cdot$$
$$\cdot(\ell'' 0, 10|L0) \sum_{\substack{m m' m''\\ \nu\nu'\chi\chi'}} (-1)^{m'} (\ell m, \tfrac{1}{2}\nu|s\mu)(\ell' m', \tfrac{1}{2}\nu'|s\mu')\cdot$$
$$\cdot(\tfrac{1}{2}\nu', 1\chi|\tfrac{1}{2}\nu)(\ell m, \ell' {-m'}|\ell'' m'')(\ell'' m'', 1\chi'|LM)(1\chi, 10|1\chi')$$

(A.12) $$I(\vec{v})\,\vec{P}\cdot\vec{v}\times(\vec{n}\times\vec{v}) = \sum_{Lm} b_3(L,m)\, Y_L^m(\vec{v})$$

where

(A.12') $$b_3(L,M) = -\sqrt{\tfrac{3}{4\pi}}\;\tfrac{1}{2}\sum_{\mu\mu'} \varrho_{\mu\mu'} \sum_{\ell\ell'} (-1)^{\ell-\ell'} T_\ell T_{\ell'}^{*}\, \hat{\ell}\hat{\ell}'\,\cdot$$
$$\cdot\Big[\sum_{\ell''\ell'''} (\ell 0, \ell' 0|\ell'' 0)(\ell'' 0, 10|\ell''' 0)(\ell''' 0, 10|L0) \sum_{\substack{m m' m''\\ m'''\nu\nu'\chi}} (-1)^{m'} (\ell m, \tfrac{1}{2}\nu|s\mu)\cdot$$
$$\cdot(\ell' m', \tfrac{1}{2}\nu'|s\mu')(\tfrac{1}{2}\nu', 1\chi|\tfrac{1}{2}\nu)(\ell m, \ell' {-m'}|\ell'' m'')(\ell'' m'', 1\chi|\ell''' m''')\cdot$$
$$\cdot(\ell''' m''', 10|LM) - (\ell 0, \ell' 0|L0) \sum_{m m'\nu\nu'} (-1)^{m'} (\ell m, \tfrac{1}{2}\nu|s\mu)(\ell' m', \tfrac{1}{2}\nu'|s\mu')\cdot$$
$$\cdot(\tfrac{1}{2}\nu', 10|\tfrac{1}{2}\nu)(\ell m, \ell' {-m'}|LM)\Big]$$

i) Calculation of $b_1(L,M)$.

To perform the summations on the magnetic quantum numbers we shall use the formula

(A.13) $$\sum_{\substack{\alpha\beta\delta\\ \varepsilon\eta\theta}} (a\alpha, b\beta|c\gamma)(d\delta, e\varepsilon|f\varphi)(g\eta, h\theta|k\chi)(a\alpha, d\delta|g\eta)(b\beta, e\varepsilon|h\theta) = $$
$$= \hat{c}\hat{f}\hat{g}\hat{h}\,(c\gamma, f\varphi|k\chi)\, X\begin{Bmatrix} a & b & c\\ d & e & f\\ g & h & k\end{Bmatrix}$$

where X is the Wigner 9-j coefficient.

The summation on the magnetic quantum numbers in (A.10') gives then:

$$(-1)^{\ell}\sqrt{2}\,\hat{s}\,\hat{L}\,\hat{\ell}''\,X\begin{Bmatrix}\ell & s & \frac{1}{2}\\ \ell' & s & \frac{1}{2}\\ \ell'' & L & 1\end{Bmatrix}(s\mu', Lm/s\mu),$$

Using now the identity:

$$\hat{a}\,\hat{d}\,\hat{h}\,\hat{k}\sum_{g}(a0,d0/g0)(h0,k0/g0)X\begin{Bmatrix}a & b & \frac{1}{2}\\ d & e & \frac{1}{2}\\ g & h & 1\end{Bmatrix}=$$

(A.14)

$$=(-1)^{d+e+k-\frac{1}{2}}\frac{1}{\sqrt{2}}(b\tfrac{1}{2},e-\tfrac{1}{2}/h0)$$

which can be proved using the expression of the X-coefficient in terms of Racah W-coefficients and (A.4), we get finally:

(A.15) $$b_1(L,M)=(-1)^{s-\frac{1}{2}}\frac{\hat{s}}{\sqrt{4\pi}}(s\tfrac{1}{2},s-\tfrac{1}{2}/L0)\sum_{\ell\ell'}T_{\ell}T_{\ell'}^{*}\sum_{\mu\mu'}\rho_{\mu\mu'}(s\mu',LM/s\mu)$$

Observing now that in (A.10') l+l'+L must be odd and remembering (3.11) and (3.12) we see that (A.15) coincides with (3.8') and (3.8") for even L and odd L respectively.

ii) Calculation of $b_2(L,M)$.

In (A.11') we first perform the sum over x'; using (A.3) we obtain:

$$\sum_{x'}(1x,10/1x')(\ell''m''1x'/LM)=$$

$$=-\sqrt{3}\sum_{f\varphi}\hat{f}\,W(11L\ell'',1f)(1x,\ell''m''/f\varphi)(10,f\varphi/LM)$$

We now perform the sum over the others magnetic quantum numbers by means of (A.13) getting

$$b_2(L,M)=i\frac{6}{\sqrt{4\pi}}\frac{\hat{s}}{\hat{L}}\sum_{\mu\mu'}\rho_{\mu\mu'}\sum_{\ell\ell'}(-1)^{\ell-\ell'}T_{\ell}T_{\ell'}^{*}\,\hat{\ell}\,\hat{\ell}'\sum_{\ell''}\hat{\ell}''(\ell 0,\ell' 0/\ell'' 0)\cdot$$

(A.16)

$$\cdot(\ell''0,10/L0)\sum_{f\varphi}\hat{f}^{2}W(11L\ell'';1f)X\begin{Bmatrix}\ell & s & 1/2\\ \ell' & s & 1/2\\ \ell'' & f & 1/2\end{Bmatrix}(s\mu',f\varphi/s\mu)(10,f\varphi/LM).$$

We must now distinguish between the case of odd-L and the case of even-L. In the former case the sum over f contains the one term f = L; furthermore we can use the identity

$$\hat{\ell}\,\hat{\ell}'\sum_{\ell''}(\ell 0,\ell' 0/\ell'' 0)(\ell'' 0,10/L0)W(11L\ell'';1L)X\begin{Bmatrix}\ell & s & 1/2\\ \ell' & s & 1/2\\ \ell'' & L & 1\end{Bmatrix}=$$

(A.17)

$$=\frac{1}{6}\frac{\hat{s}^{2}}{\hat{L}\sqrt{L(L+1)}}(s\tfrac{1}{2},s-\tfrac{1}{2}/L0)$$

so that for odd-L, being l' = l and remembering (3.12"), we easily obtain (3.9').

For even L we have f = l" = L ± 1. Using now the identity

(A.18) $$\hat{\ell}\,\hat{\ell}'(\ell 0,\ell' 0/\ell'' 0)X\begin{Bmatrix}\ell & s & 1/2\\ \ell' & s & 1/2\\ \ell'' & \ell'' & 1\end{Bmatrix}=\frac{1}{\sqrt{6}}\frac{\hat{s}^{2}}{\hat{\ell}''\sqrt{\ell''(\ell''+1)}}(s\tfrac{1}{2},s-\tfrac{1}{2}/\ell'' 0)$$

and the explicit expression

(A.19) $$(\ell''0,10/L0)W(1/L\ell'';1\ell'') = -\frac{1}{\sqrt{6}}\frac{1}{2\ell''+1}F_{\ell''}$$

where

(A.19') $$F_{\ell''} = \begin{cases} \sqrt{\ell''} & \text{for } l'' = L-1 \\ \sqrt{\ell''+1} & \text{for } l'' = L+1 \end{cases}$$

and using the definition (3.12') we immediately obtain (3.9").

iii) Calculation of $b_3(L,M)$.

Performing in (A.12') the summations on the magnetic quantum numbers making use of (A.13) we obtain:

(A.20) $$b_3(L,M) = \sqrt{\frac{6}{4\pi}}\frac{\hat{s}}{2}\sum_{\mu\mu'}\rho_{\mu\mu'}\sum_{\ell\ell'}(-1)^{\ell-\ell'}T_\ell T^*_{\ell'}\hat{\ell}\hat{\ell}'\Big[\sum_{\ell''\ell'''}(-1)^{\ell'+\ell'''}\hat{\ell}''\hat{\ell}'''(\ell 0,\ell'0/\ell''0)\cdot$$
$$\cdot(\ell''0,10/\ell'''0)(\ell'''0,10/L0)(s\mu',\ell'''M/s\mu)(\ell'''M,10/LM)X\begin{Bmatrix}\ell & s & 1/2\\ \ell' & s & 1/2\\ \ell'' & \ell''' & 1\end{Bmatrix}+$$
$$+(-1)^{\ell'}\hat{L}(\ell 0,\ell'0/L0)\sum_f\hat{f}(s\mu',fM/s\mu)(10,LM/fM)X\begin{Bmatrix}\ell & s & 1/2\\ \ell' & s & 1/2\\ L & f & 1\end{Bmatrix}\Big].$$

In the first term in the square brackets we perform the sum mation over l" using the identity (A.14) and also we change l"' in f so that $b_3(L,M)$ can be written:

(A.21) $$b_3(L,M) = \frac{\hat{s}}{\sqrt{4\pi}}\sum_{\mu\mu'}\rho_{\mu\mu'}\sum_{\ell\ell'}(-1)^{\ell'}T_\ell T^*_{\ell'}\sum_f(10,LM/fM)(s\mu',fM/s\mu)D(f,L)$$

where

(A.21') $$D(f,L) = \sqrt{6}\,\hat{\ell}\hat{\ell}'\hat{f}(\ell 0,\ell'0/L0)X\begin{Bmatrix}\ell & s & 1/2\\ \ell' & s & 1/2\\ L & f & 1\end{Bmatrix} -$$
$$-(-1)^{L+\ell+s+\frac{1}{2}}(f0,10/L0)(s\tfrac{1}{2},s-\tfrac{1}{2}/f0).$$

Let us consider first the odd-L case. For f = L±1 is D(f,L)=0; it remains the term with f = L which, being l=l'±1 and using (A.18) and (3.12'), gives rise to (3.10').
For even L, D(f,L) is zero if f = L; for f = L ± 1 is:

$$D(f,f-1) = \frac{2s+1}{\sqrt{f(2f+1)}}(s\tfrac{1}{2},s-\tfrac{1}{2}/f0)$$
$$D(f,f+1) = \frac{2s+1}{\sqrt{(f+1)(2f+1)}}(s\tfrac{1}{2},s-\tfrac{1}{2}/f0)$$

Putting these expressions in (A.21) and using (3.12") we ob tain (3.10").

Bibliography.

(1) - M. Peshkin, Phys. Rev. 129, 1864 (1963).

(2) - A. Bohr; Nuclear Phys. 10, 486 (1959).

(3) - P. Eberhard and M.L. Good, Phys. Rev. 120, 1442 (1960).

(4) - T.D. Lee and C.N. Yang, Phys. Rev. 109, 1755 (1959).

(5) - L. Durand, L.F. Landovitz and J. Leitner, Phys. Rev. 112, 273 (1958).

(6) - R. Spitzer and H.P. Stapp, University of California Radiation Laboratory, Report UCRL-3796 (unpublished); Phys. Rev. 109, 540 (1958).

(7) - R. Gatto and H.P. Stapp, Phys. Rev. 121, 1553 (1961).

(8) - R.H. Capps, Phys. Rev. 122, 929 (1961).

(9) - M. Ademollo and R. Gatto, Nuovo Cimento, to be published.

(10)- N. Byers and S. Fenster, Phys. Rev. Letters 11, 52 (1963).

(11)- R.K. Adair, Phys. Rev. 100, 1540 (1955).

TABLE I - Numerical values coefficients $C^{(s)}_{\mu,\mu+M}(L)$

I a: $s = \frac{1}{2}$

L	M	μ	
		$+\frac{1}{2}$	$-\frac{1}{2}$
0	0	1.7725	1.7725
1	0	3.0700	-3.0700

I b: $s = \frac{3}{2}$

L	M	μ			
		$+\frac{3}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$
0	0	0.8862	0.8862	0.8862	0.8862
1	0	4.6050	1.5350	-1.5350	-4.6050
2	0	-1.9817	1.9817	1.9817	-1.9817
2	2	0.0000	0.0000	-2.8025	-2.8025
3	0	-0.7816	2.3447	-2.3447	0.7816
3	2	0.0000	0.0000	-2.4716	2.4716

Ic: $s = \frac{5}{2}$

L	M	μ					
		+5/2	+3/2	+1/2	-1/2	-3/2	-5/2
0	0	0.5908	0.5908	0.5908	0.5908	0.5908	0.5908
1	0	5.1166	3.0700	1.0233	-1.0233	-3.0700	-5.1166
2	0	-1.6514	0.8303	1.3211	1.3211	0.3303	-1.6514
2	2	0.0000	0.0000	-1.2792	-1.7162	-1.7162	-1.2792
3	0	-1.9540	2.7355	1.5632	-1.5632	-2.7355	1.9540
3	2	0.0000	0.0000	-3.3843	-1.5135	1.5135	3.3843
4	0	0.8862	-2.6587	1.7725	1.7725	-2.6587	0.8862
4	2	0.0000	0.0000	2.6587	-1.9817	-1.9817	2.6587
4	4	0.0000	0.0000	0.0000	0.0000	3.3160	3.3160
5	0	0.1960	-0.9798	1.9595	-1.9595	0.9798	-0.1960
5	2	0.0000	0.0000	0.8980	-2.0079	2.0079	-0.8980
5	4	0.0000	0.0000	0.0000	0.0000	2.1996	-2.1996